Introduction to (Convolutional) Neural Networks

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Syllabus

1. Motivation and Definition
2. Universal Approximation
3. Backpropagation
4. Stochastic Gradient Descent
5. The Basic Recipe
6. Going Deep
7. Convolutional Neural Networks
8. What I didn't tell you

1 Motivation and Definition

Which Method to Choose?

We have seen linear Regression, kernel regression, regularization, K-PCA, K-SVM, ... and there exist a zillion other methods.

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Is there a universally best method?

No Free Lunch Theorem

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"No Free Lunch" Theorem [Wolpert(1996)], Informal Version

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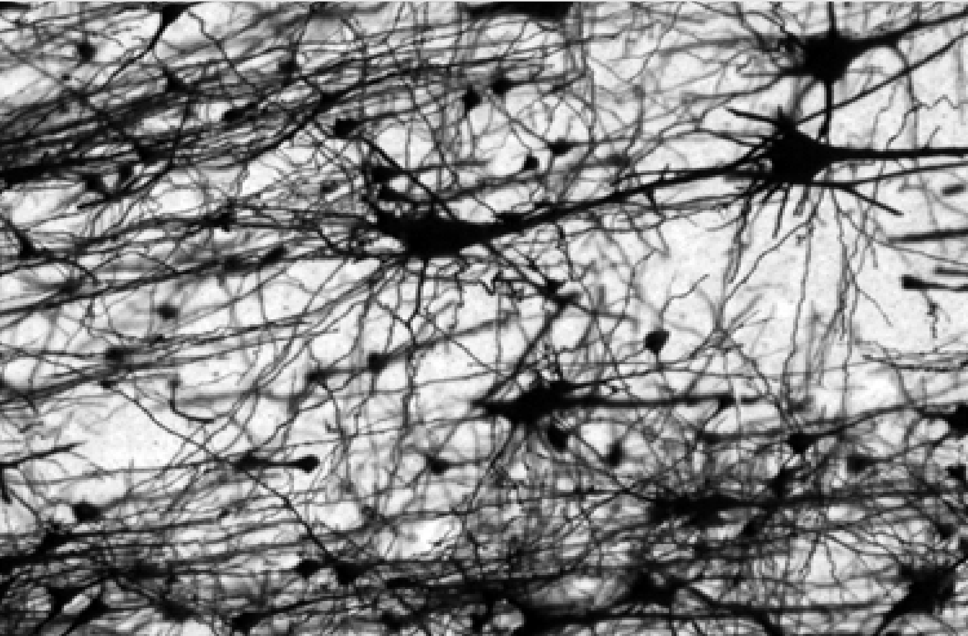
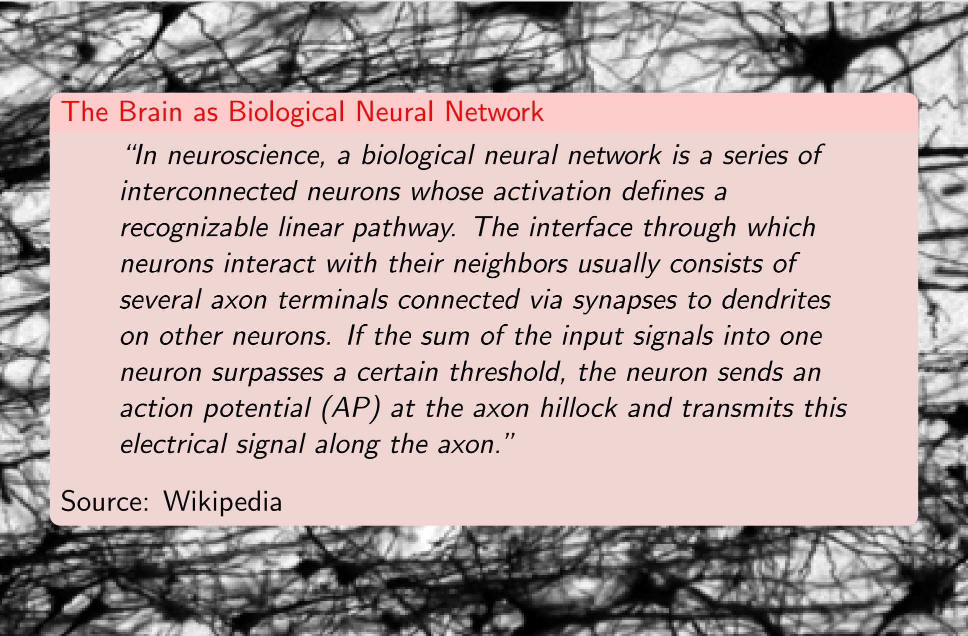
We want our algorithm to reproduce the artificial categories produced by our brain – so let's build a hypothesis class that mimicks our thinking!

Neuroscience



Neuroscience

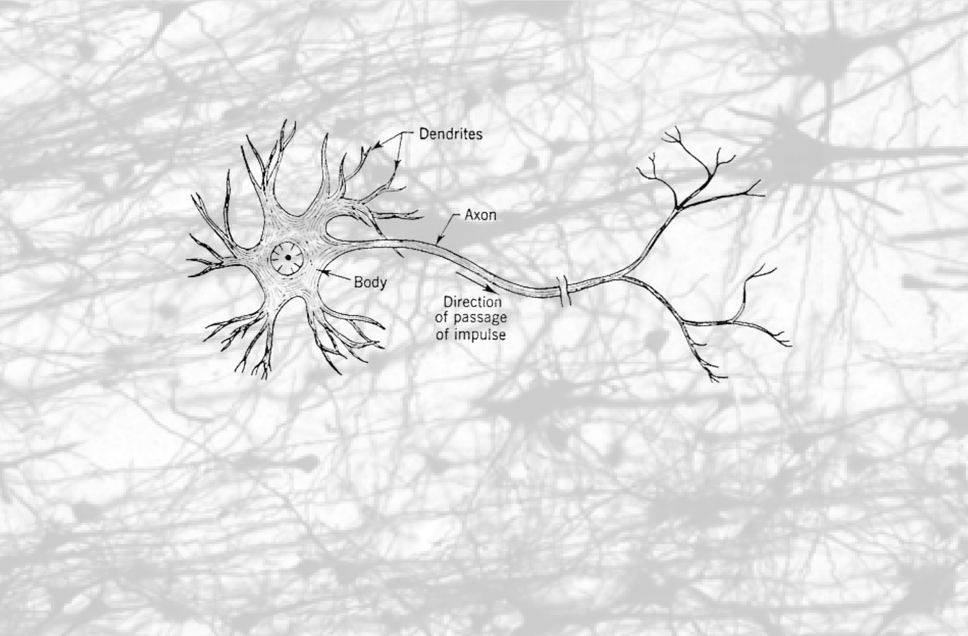


The Brain as Biological Neural Network

> *"In neuroscience, a biological neural network is a series of interconnected neurons whose activation defines a recognizable linear pathway. The interface through which neurons interact with their neighbors usually consists of several axon terminals connected via synapses to dendrites on other neurons. If the sum of the input signals into one neuron surpasses a certain threshold, the neuron sends an action potential (AP) at the axon hillock and transmits this electrical signal along the axon."*

Source: Wikipedia

Neurons



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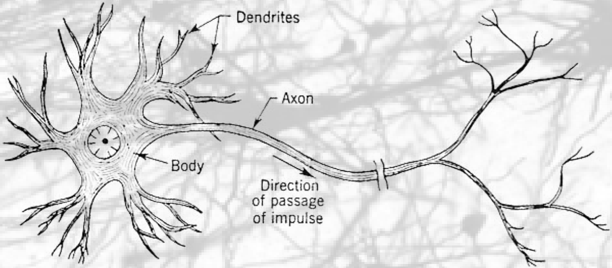
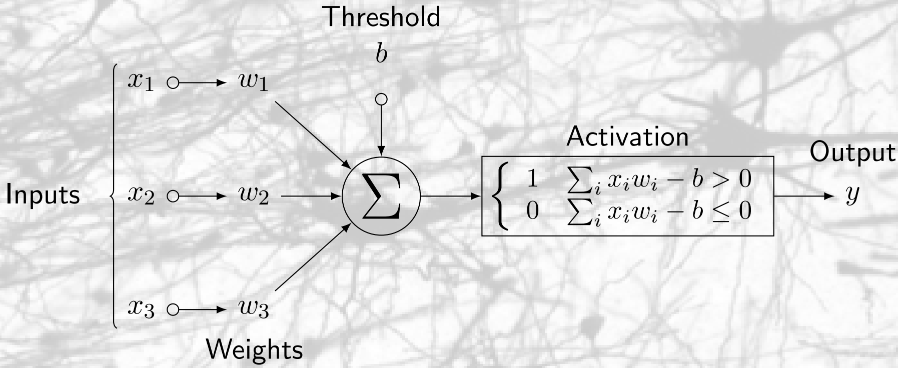


recall: *"If the sum of the input signals into one neuron surpasses a certain threshold, [...] the neuron transmits this [...] signal [...]."*

Artificial Neurons



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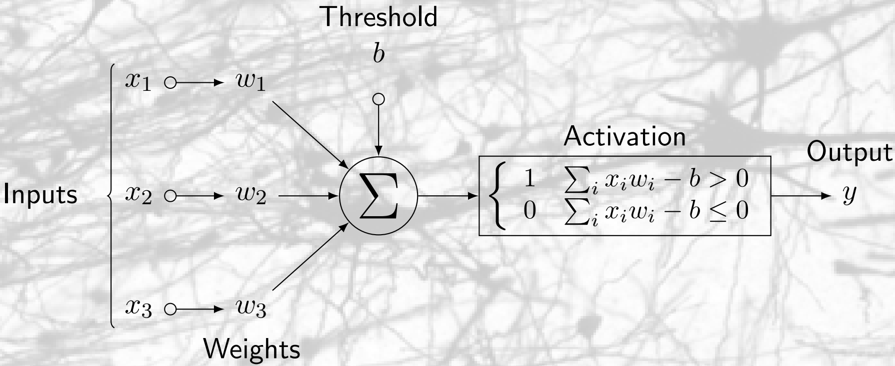


Artificial Neuron

An *artificial neuron* with *weights* $w_1, \dots, w_s$, *bias* b and *activation function* $\sigma : \mathbb{R} \to \mathbb{R}$ is defined as the function

$$f(x_1, \dots, x_s) = \sigma\left(\sum_{i=1}^{s} x_i w_i - b\right).$$

Activation Functions

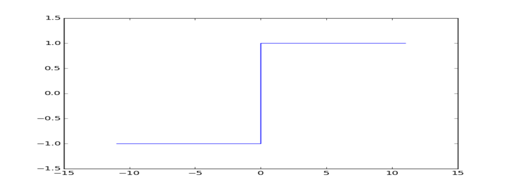
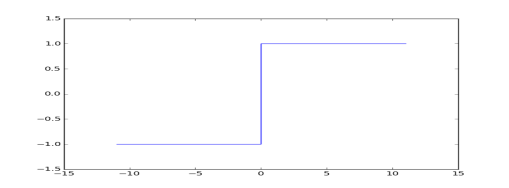


Figure: Heaviside activation function (as in biological motivation)

Activation Functions



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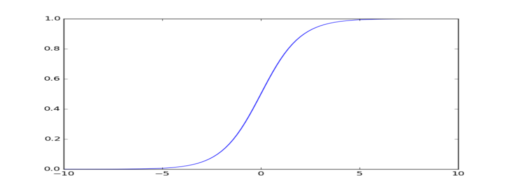
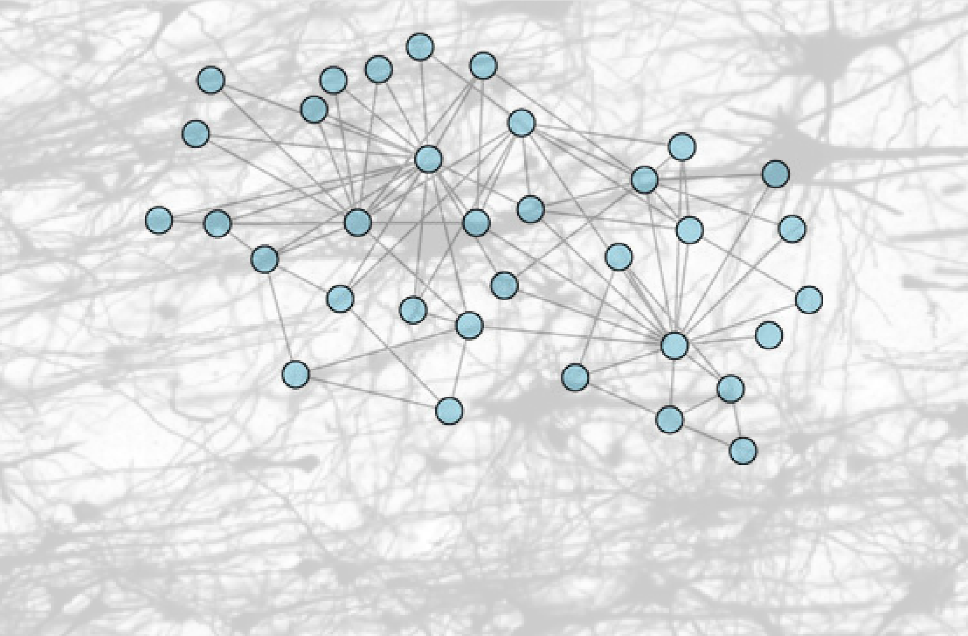


Figure: Sigmoid activation function $\sigma(x) = \frac{1}{1+e^{-x}}$

Artificial Neural Networks



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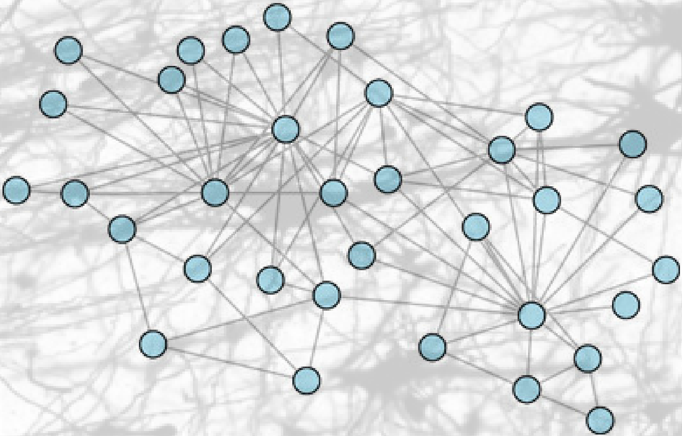

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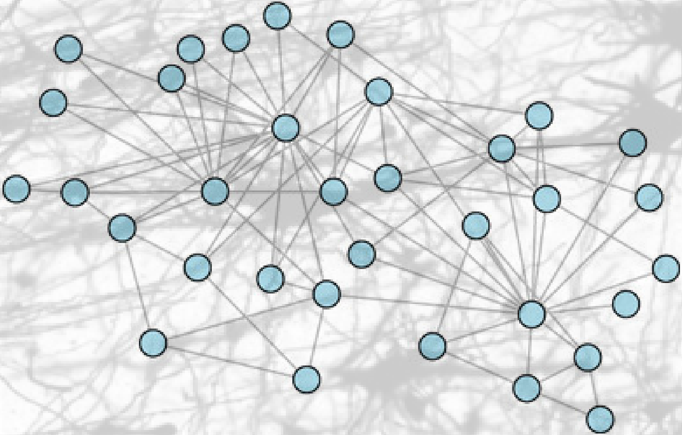

- Artificial neural networks consist of a graph, connecting artificial neurons!
- Dynamics difficult to model, due to loops, etc...

Artificial Feedforward Neural Networks

Use directed, acyclic graph!

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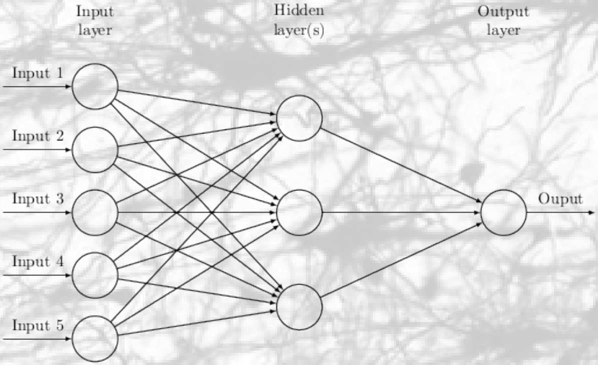

Artificial Feedforward Neural Networks

Definition

Let $L, d, N_1, \ldots, N_L \in \mathbb{N}$. A map $\Phi : \mathbb{R}^d \to \mathbb{R}^{N_L}$ given by

$$\Phi(x) = A_L \sigma \left(A_{L-1} \sigma \left(\ldots \sigma \left(A_1(x)\right)\right)\right), \quad x \in \mathbb{R}^d,$$

is called a *neural network*. It is composed of affine linear maps $A_\ell : \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{\mathbb{N}_\ell}$, $1 \le \ell \le L$ (where $N_0 = d$), and non-linear functions—often referred to as *activation function*—σ acting component-wise. Here, d is the *dimension of the input layer*, L denotes the *number of layers*, $N_1, \ldots, N_{L-1}$ stands for the *dimensions of the $L-1$ hidden layers*, and N_L is the *dimension of the output layer*.

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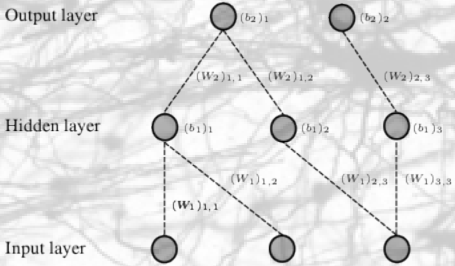
An affine map $A : \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{N_\ell}$ is given by $x \mapsto Wx + b$ with *weight matrix* $W \in \mathbb{R}^{N_{\ell-1} \times N_\ell}$ and *bias vector* $b \in \mathbb{R}^{N_\ell}$.

Artificial Feedforward Neural Networks



$$W_2 = \begin{pmatrix} (W_2)_{1,1} & (W_2)_{1,2} & 0 \\ 0 & 0 & (W_2)_{2,3} \end{pmatrix}$$

$$W_1 = \begin{pmatrix} (W_1)_{1,1} & (W_1)_{1,2} & 0 \\ 0 & 0 & (W_1)_{2,3} \\ 0 & 0 & (W_1)_{3,3} \end{pmatrix}$$

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Artificial feedforward neural networks constitute a mathematically and computationally convenient but very simplistic mathematical construct which is inspired by our understanding of how the brain works.

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- **"Deep Learning":** Neural network learning with neural networks consisting of many (e.g., ≥ 3) layers.

2 Universal Approximation

Approximation Question

Main Approximation Problem

can every (continuous, or measurable) function $f : \mathbb{R}^d \to \mathbb{R}^{N_L}$ be arbitrarily well approximated by a neural network, provided that we choose $N_1, \ldots, N_{L-1}, L$ large enough?

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Surely not!

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Surely not! Suppose that σ is a polynomial of degree r. Then $\sigma(Ax)$ is a polynomial of degree $\leq r$ for all affine maps A and therefore any neural network with activation function σ will be a polynomial of degree $\leq r$.

Approximation Question

Main Approximation Problem

Under which conditions on the activation function σ can every (continuous, or measurable) function $f : \mathbb{R}^d \to \mathbb{R}^{N_L}$ be arbitrarily well approximated by a neural network, provided that we choose $N_1, \ldots, N_{L-1}, L$ large enough?

Universal Approximation Theorem

Theorem

Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ continuous is not a polynomial and fix $d \geq 1, L \geq 2, N_L \geq 1 \in \mathbb{N}$ and a compact subset $K \subset \mathbb{R}^d$. Then for any continuous $f : \mathbb{R}^d \to \mathbb{R}^{N_l}$ and any $\varepsilon > 0$ there exist $N_1, \ldots, N_{L-1} \in \mathbb{N}$ and affine linear maps $A_\ell : \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{N_\ell}$, $1 \leq \ell \leq L$ such that the neural network

$$\Phi(x) = A_L \sigma \left(A_{L-1} \sigma \left(\ldots \sigma \left(A_1(x) \right) \right) \right), \quad x \in \mathbb{R}^d,$$

approximates f to within accuracy ε, i.e.,

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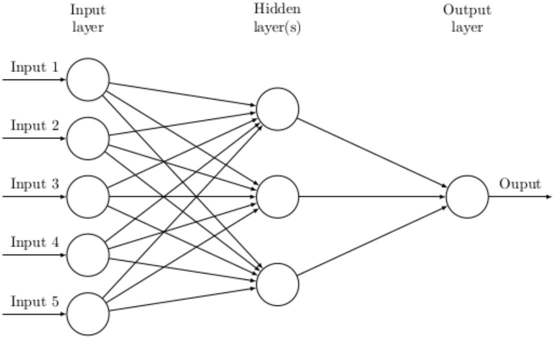
Neural networks are "universal approximators" and one hidden layer is enough if the number of nodes is sufficient!

Skip Proof

Proof of the Universal Approximation Theorem

For simplicity we only the case of one hidden layer, e.g., $L = 2$ and one output neuron, e.g., $N_L = 1$:



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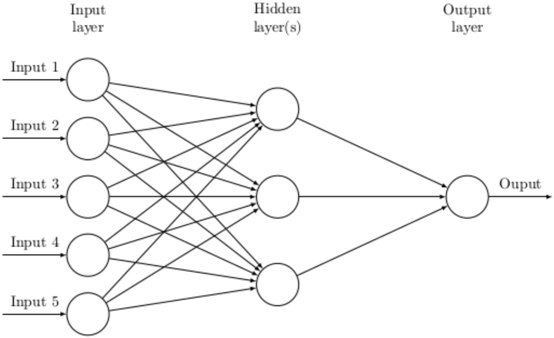

$$\Phi(x) = \sum_{i=1}^{N_1} c_i \sigma(w_i \cdot x - b_i), \quad w_i \in \mathbb{R}^d, \; c_i, b_i \in \mathbb{R}.$$

Proof of the Universal Approximation Theorem

We will show the following.

Theorem

For $d \in \mathbb{N}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ continuous consider

$$\mathcal{R}(\sigma, d) := \text{span}\left\{\sigma(w \cdot x - b) : \; w \in \mathbb{R}^d, \; b \in \mathbb{R}\right\}.$$

Then $\mathcal{R}(\sigma, d)$ is dense in $C(\mathbb{R}^d)$ if and only if σ is not a polynomial.

Proof for $d = 1$ and σ smooth

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- Stone-Weierstrass Theorem yields the result.

General d

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- Note that the functions

$$\text{span}\{g(w \cdot x - b) : \; w \in \mathbb{R}^d, \; b \in \mathbb{R}, g \in C(\mathbb{R}) \text{ arbitrary}\},$$

are dense in $C(\mathbb{R}^d)$ (just take g as $\sin(w \cdot x), \cos(w \cdot x)$ just as in the Fourier series case).

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- First approximate $f \in C(\mathbb{R}^d)$ by

$$\sum_{i=1}^{N} d_i g_i(v_i \cdot x - e_i), \quad v_i \in \mathbb{R}^d, \ d_i, e_i \in \mathbb{R}, \ g_i \in C(\mathbb{R}).$$

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- Then apply our univariate result to approximate the univariate functions $t \mapsto g_i(t - e_i)$ using neural networks.

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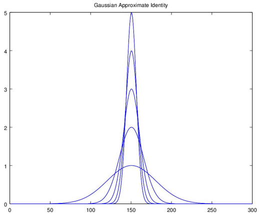
- pick family $(g_\varepsilon)_{\varepsilon>0}$ of mollifiers, i.e.

$$\lim_{\varepsilon\to 0} \sigma * g_\varepsilon \to \sigma$$

uniformly on compacta.



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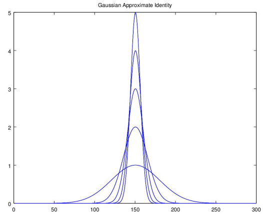

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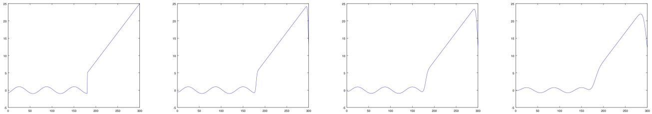

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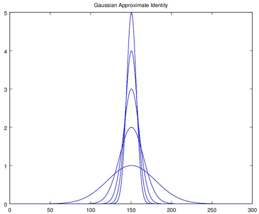

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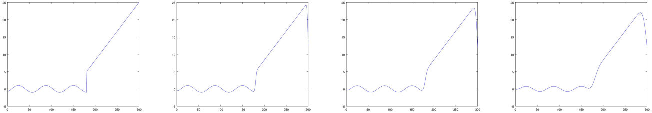

□

3 Backpropagation

Regression/Classification with Neural Networks

Neural Network Hypothesis Class

Given $d, L, N_1, \ldots, N_L$ and σ define the associated hypothesis class

$$\mathcal{H}_{[d,N_1,\ldots,N_L],\sigma} := \left\{ A_L\sigma\left(A_{L-1}\sigma\left(\ldots\sigma\left(A_1(x)\right)\right)\right) : \ A_\ell : \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{N_\ell} \text{ affine linear } \right\}.$$

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Typical Regression/Classification Task

Given data $\mathbf{z} = ((x_i, y_i))_{i=1}^m \subset \mathbb{R}^d \times \mathbb{R}^{N_L}$, find the empirical regression function

$$f_{\mathbf{z}} \in \operatorname{argmin}_{f \in \mathcal{H}_{[d,N_1,\dots,N_L],\sigma}} \sum_{i=1}^m \mathcal{L}(f, x_i, y_i),$$

where $\mathcal{L} : C(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^{N_L} \to \mathbb{R}_+$ is the *loss function* (in least squares problems we have $\mathcal{L}(f, x, y) = |f(x) - y|^2$).

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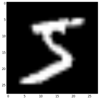

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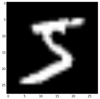

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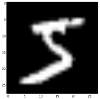

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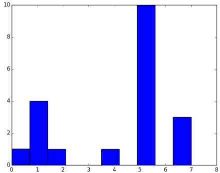

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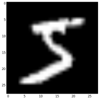

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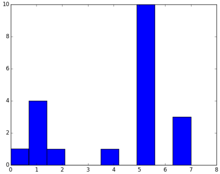

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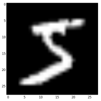

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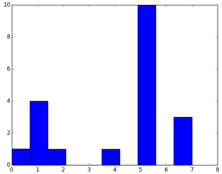

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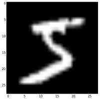

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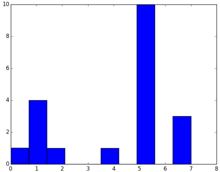

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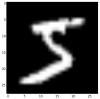

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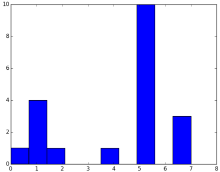

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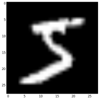

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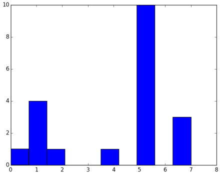

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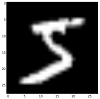

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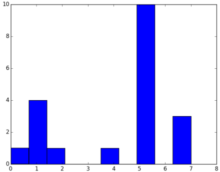

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Non-linear, non-convex

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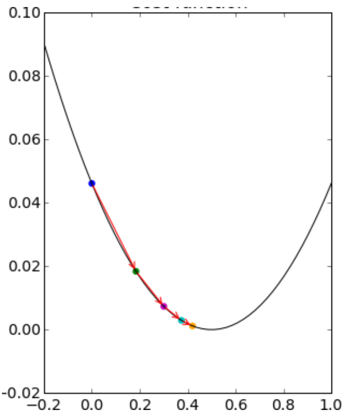

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Skip Derivation

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For simplicity suppose that $\mathcal{L}(f, x, y) = (f(x) - y)^2$, so that

$$\frac{\partial \mathcal{L}(f,x,y)}{\partial (W_\ell)_{i,j}} = 2 \cdot (f(x) - y)^T \cdot \frac{\partial f(x)}{\partial (W_\ell)_{i,j}},$$

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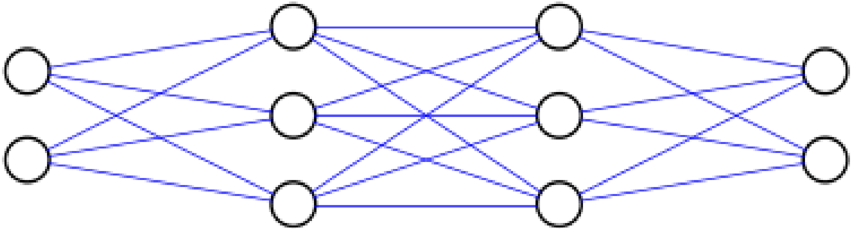

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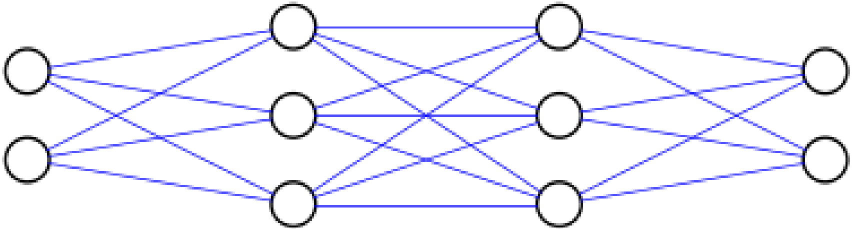

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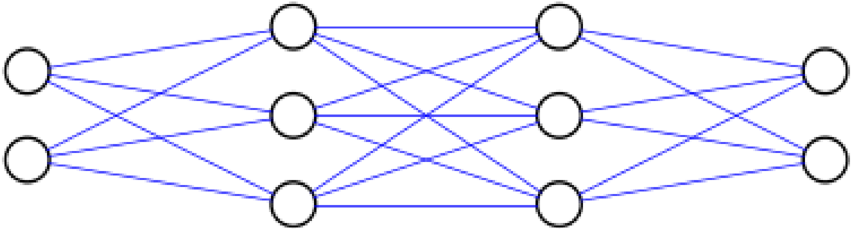

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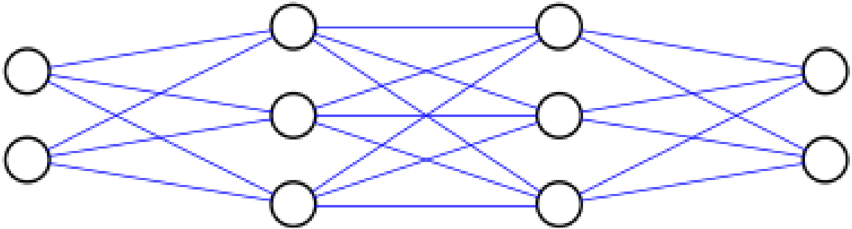

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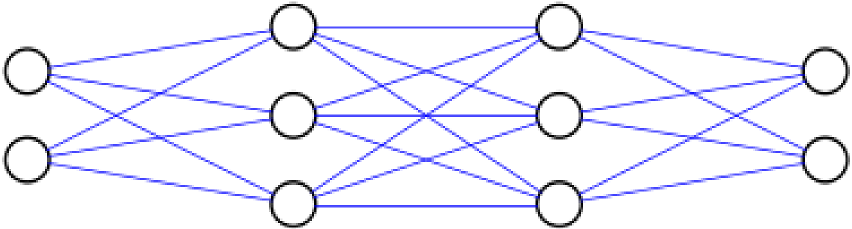

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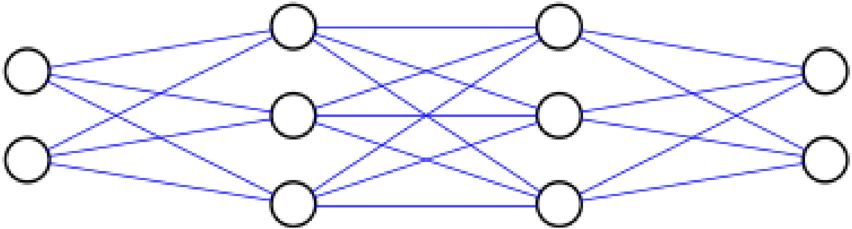

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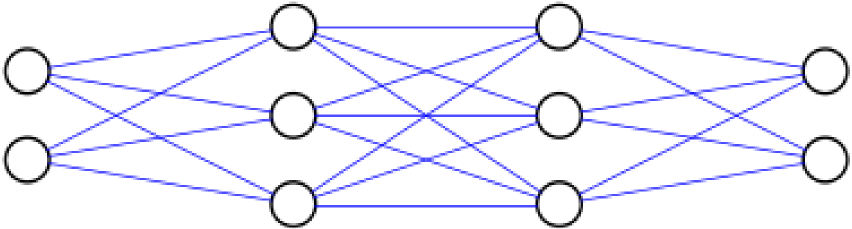

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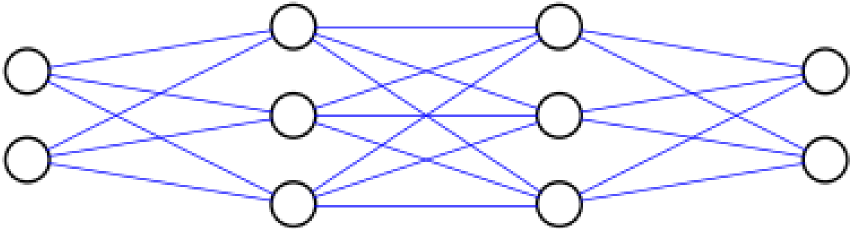

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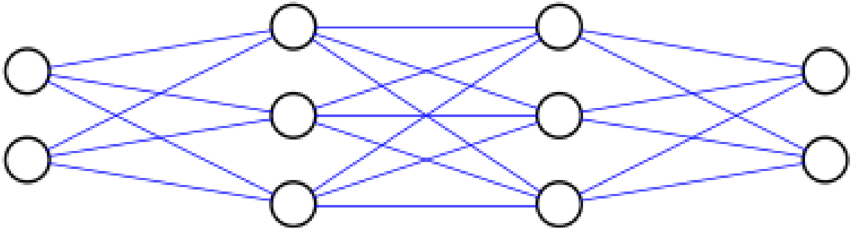

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$$\frac{\partial (z_3)_2}{\partial (W_3)_{1,2}} = \frac{\partial}{\partial (W_3)_{1,2}} \left((W_3)_{2,1}(a_2)_1 + (W_3)_{2,2}(a_2)_2 + (W_3)_{2,3}(a_2)_3\right) = 0$$

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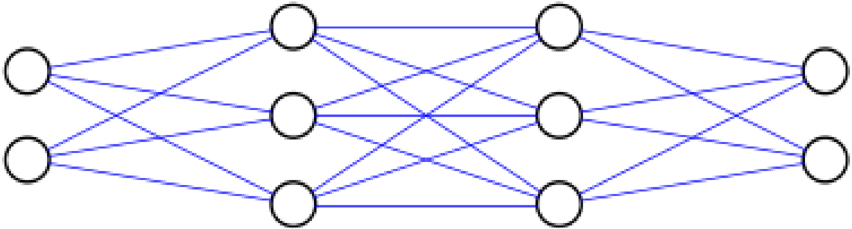

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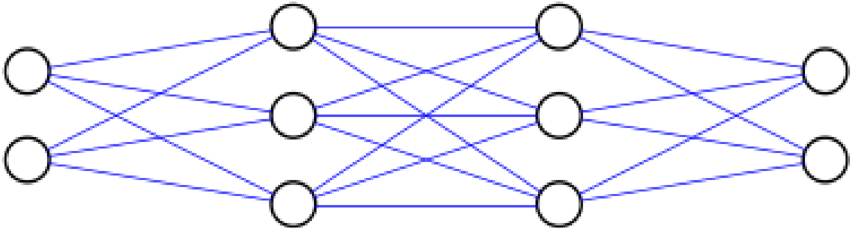

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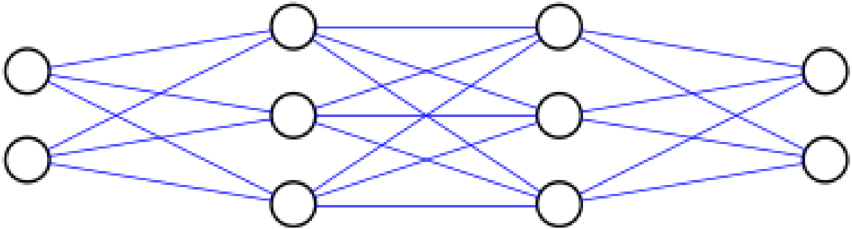

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$$\frac{\partial \Phi(x)}{\partial W_3} = \begin{pmatrix} \begin{pmatrix} (a_2)_1 \\ 0 \\ 0 \\ (a_2)_1 \end{pmatrix} \begin{pmatrix} (a_2)_2 \\ 0 \\ 0 \\ (a_2)_2 \end{pmatrix} \begin{pmatrix} (a_2)_3 \\ 0 \\ 0 \\ (a_2)_3 \end{pmatrix} \end{pmatrix}$$

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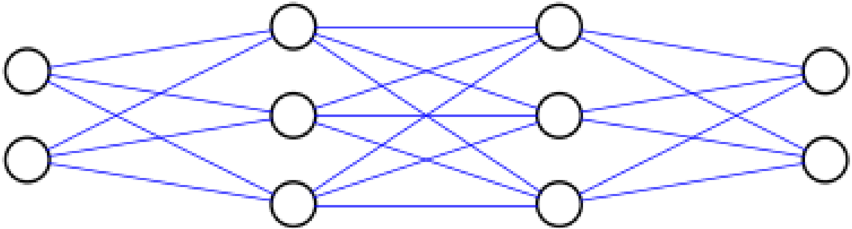
$$b_3 = \begin{pmatrix} (b_3)_1 \\ (b_3)_2 \end{pmatrix}$$

$$\frac{\partial \Phi(x)}{\partial b_3} = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$



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Backprop: Last Layer

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- $\frac{\partial \mathcal{L}(f,x,y)}{\partial (W_L)_{i,j}} = 2 \cdot (f(x) - y)^T \cdot \frac{\partial f(x)}{\partial (W_L)_{i,j}}$,
  $\frac{\partial \mathcal{L}(f,x,y)}{\partial (b_L)_i} = 2 \cdot (f(x) - y)^T \cdot \frac{\partial f(x)}{\partial (b_L)_i}$.

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- Let $f(x) = W_L \sigma(W_{L-1}(\dots) + b_{L-1}) + b_L$. It follows that
  - $\frac{\partial f(x)}{\partial (W_L)_{i,j}} = (0, \dots, \underbrace{\sigma(W_{L-1}(\dots) + b_{L-1})_j}_{i}, \dots, 0)^T$
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  - $2(f(x) - y)^T \frac{\partial f(x)}{\partial (W_L)_{i,j}} = 2(f(x) - y)_i \sigma(W_{L-1}(\dots) + b_{L-1})_j$,
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  - $2(f(x) - y)^T \frac{\partial f(x)}{\partial (b_L)_i} = 2(f(x) - y)_i$

In matrix notation:

- $\frac{\partial \mathcal{L}(f,x,y)}{\partial W_L} = \underbrace{2(f(x) - y)}_{\delta_L} (\underbrace{\sigma(\overbrace{W_{L-1}(\dots) + b_{L-1}}^{z_{L-1}})}_{a_{L-1}})^T$,
- $\frac{\partial \mathcal{L}(f,x,y)}{\partial b_L} = 2(f(x) - y)$.

Backprop: Second-to-last Layer

- Define $a_{\ell+1} = \sigma(z_{\ell+1})$ where $z_{\ell+1} = W_{\ell+1} a_\ell + b_{\ell+1}$, $a_0 = x$, $f(x) = z_L$.

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- Then, use chain rule:

$$\frac{\partial \mathcal{L}(f,x,y)}{\partial W_{L-1}} = \frac{\partial \mathcal{L}(f,x,y)}{a_{L-1}} \cdot \frac{\partial a_{L-1}}{\partial W_{L-1}} = 2(f(x) - y)^T \cdot W_L \cdot \frac{\partial a_{L-1}}{\partial W_{L-1}}.$$

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same as before!

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$$= \underbrace{\mathrm{diag}(\sigma'(z_{L-1})) \cdot W_L^T \cdot 2(f(x)-y)}_{\delta_{L-1}} \cdot a_{L-2}^T.$$

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- Similar arguments yield $\frac{\partial \mathcal{L}(f,x,y)}{\partial b_{L-1}} = \delta_{L-1}$.

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3. Then $\frac{\partial \mathcal{L}(f,x,y)}{\partial b_L} = \delta_L$ and $\frac{\partial \mathcal{L}(f,x,y)}{\partial W_L} = \delta_L \cdot a_{L-1}^T$.

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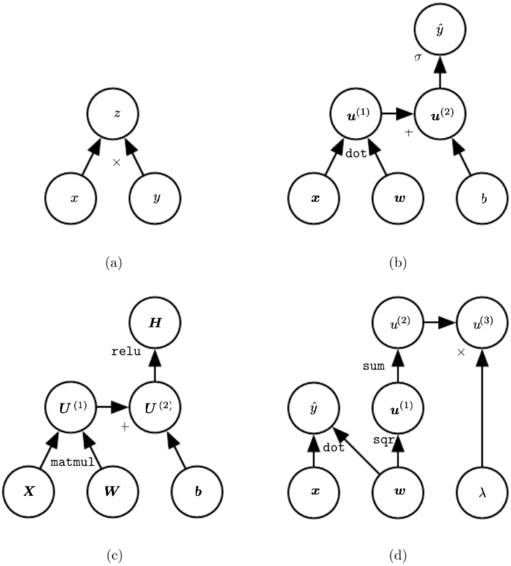
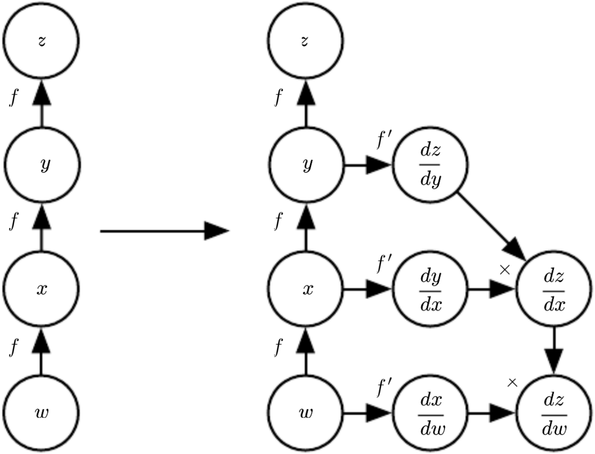
5. **return** $\frac{\partial \mathcal{L}(f,x,y)}{\partial b_\ell}, \ \frac{\partial \mathcal{L}(f,x,y)}{\partial W_\ell}$, $l = 1, \dots, L$.

Computational Graphs



Automatic Differentiation



4 Stochastic Gradient Descent

The Complexity of Gradient Descent

Recall that one gradient descent step requires the calculation of

$$\sum_{i=1}^{m} \nabla_{((W_\ell, b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_i, y_i).$$

and each of the summands requires one backpropagation run.

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Thus, the total complexity of one gradient descent step is equal to

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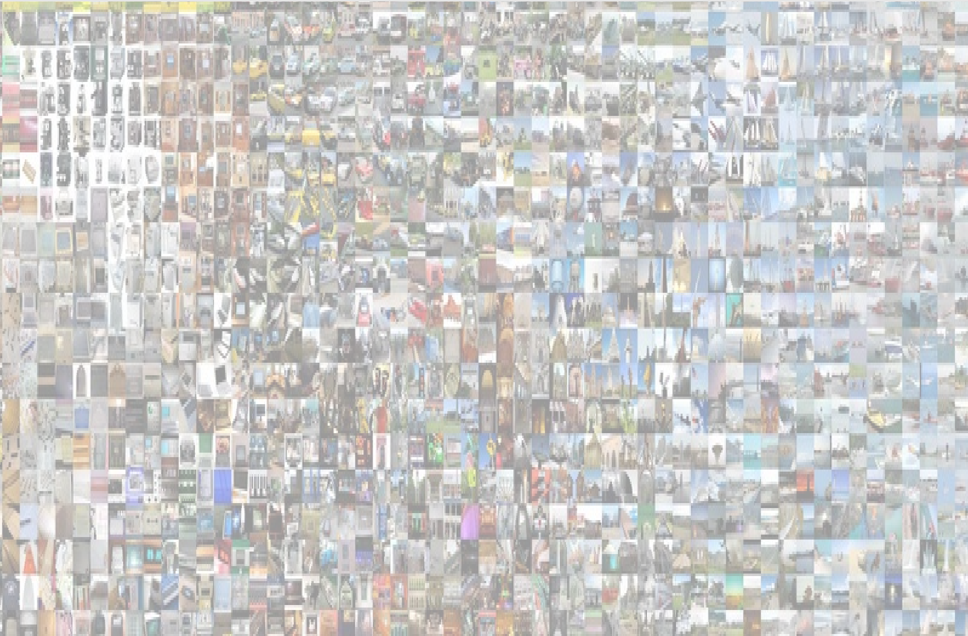
The complexity of backprop is asymptotically equal to the number of DOFs of the network:

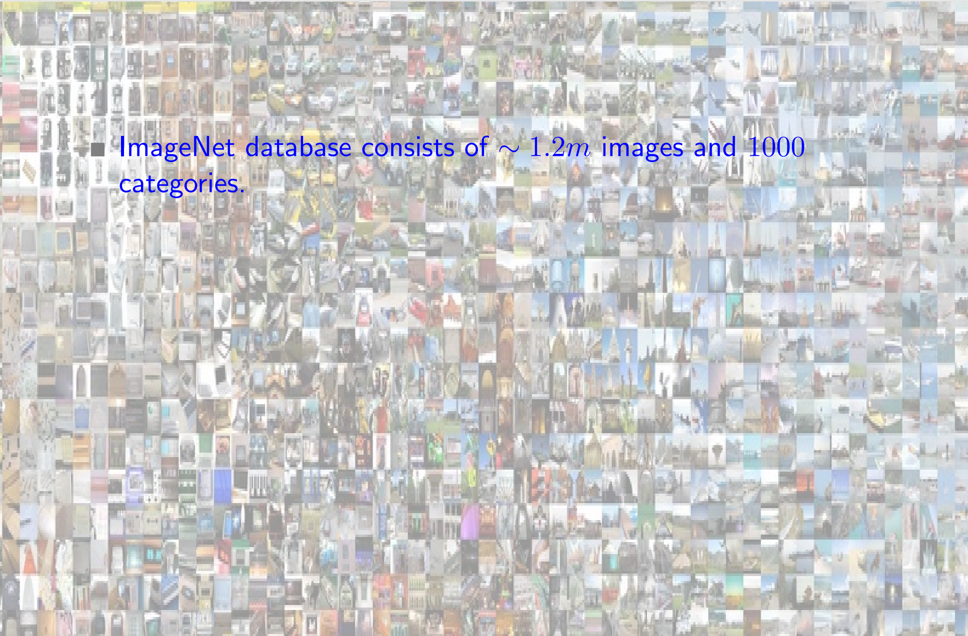
$$\text{complexity}(backprop) \sim \sum_{\ell=1}^{L} N_{\ell-1} \times N_\ell + N_\ell.$$

An Example



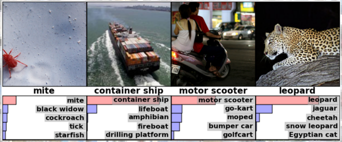
An Example

- ImageNet database consists of $\sim 1.2m$ images and 1000 categories.



An Example

- ImageNet database consists of $\sim 1.2m$ images and 1000 categories.
- AlexNet, neural network with $\sim 160m$ DOFs is one of the most successful annotation methods



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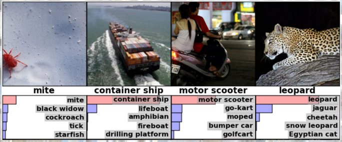

One step of gradient descent requires $\sim 2 * 10^{14}$ flops (and memory units)!!

Stochastic Gradient Descent (SGD)

Approximate

$$\sum_{i=1}^{m} \nabla_{((W_\ell, b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_i, y_i)$$

by

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for some i^* chosen *uniformly at random* from $\{1, \ldots, m\}$.

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In expectation we have

$$\mathbb{E} \nabla_{((W_\ell, b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_{i^*}, y_{i^*}) = \frac{1}{m} \sum_{i=1}^{m} \nabla_{((W_\ell, b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_i, y_i)$$

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Goal: Find stationary point of function $F = \sum_{i=1}^{m} F_i : \mathbb{R}^N \to \mathbb{R}$.

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Typical Behavior

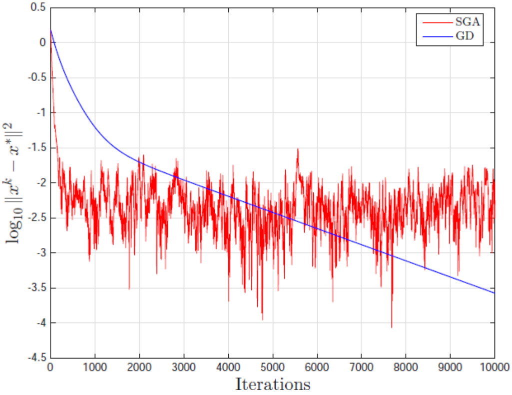

Figure: Comparison btw. GD and SGD. m steps of SGD are counted as one iteration.

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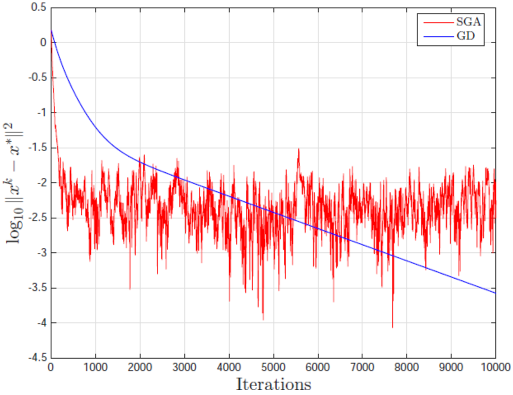

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Initially very fast convergence, followed by stagnation!

Minibatch SGD


For every $\{i_1^*, \ldots, i_K^*\} \subset \{1, \ldots, m\}$ chosen uniformly at random, it holds that

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- $K = 1 \rightsquigarrow$ SGD
- $K > 1 \rightsquigarrow$ Minibatch SGD with batchsize K.

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- The *sample mean* $\frac{1}{K}\sum_{l=1}^{k} \nabla_{((W_\ell,b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_{i_l^*}, y_{i_l^*})$ is itself a random variable that has expected value $\frac{1}{m}\sum_{i=1}^{m} \nabla_{((W_\ell,b_\ell))_{\ell=1}^{L}} \mathcal{L}(f, x_i, y_i)$.

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Common batchsize for large models: $K=16,32$.

5 The Basic Recipe

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Now let's try classifying handwritten digits!

Results

MNIST dataset, 30 epochs, learning rate $\eta = 3.0$, minibatch size $K = 10$, training set size $m = 50000$, test set size $= 10000$.

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Deep learning might not help after all...

6 Going Deep (?)

Problems with Deep Networks

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- Vanishing/Exploding Gradient Problem

Dealing with Overfitting: Regularization

Rather than minimizing

$$\sum_{i=1}^{m} \mathcal{L}(f, x_i, y_i),$$

minimize

$$\sum_{i=1}^{m} \mathcal{L}(f, x_i, y_i) + \lambda \Omega((W_\ell)_{\ell=1}^{L}),$$

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Gradient update has to be augmented by

$$\lambda \cdot \frac{\partial}{\partial (W_\ell)_{i,j}} \Omega((W_\ell)_{\ell=1}^{L}) = \lambda \cdot p \cdot |(W_\ell)_{i,j}|^{p-1} \cdot \mathsf{sgn}((W_\ell)_{i,j})$$

Sparsity-Promoting Regularization

Since

$$\lim_{p \to 0} \sum_{l,i,j} |(W_\ell)_{i,j}|^p = \#\text{nonzero weights},$$

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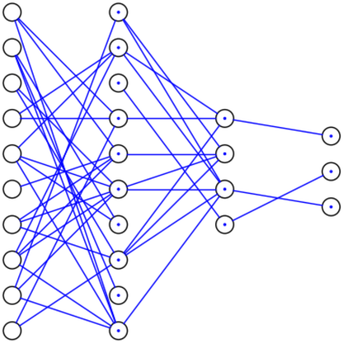

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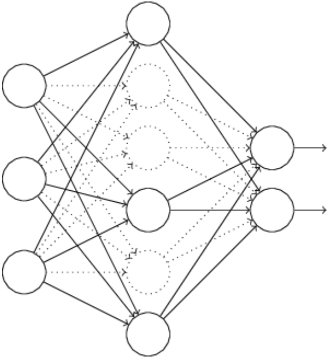

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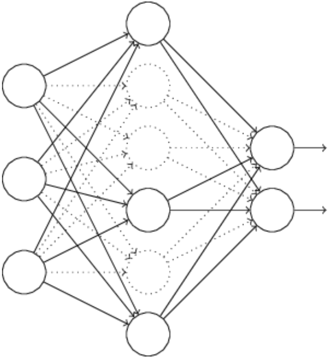

After training, multiply all weights with p.

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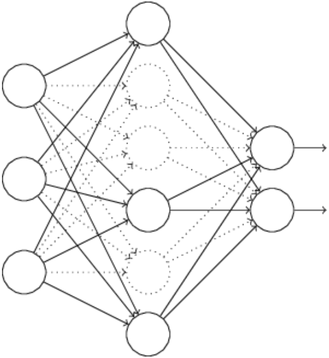
Final output is "average" over many sparse network models.



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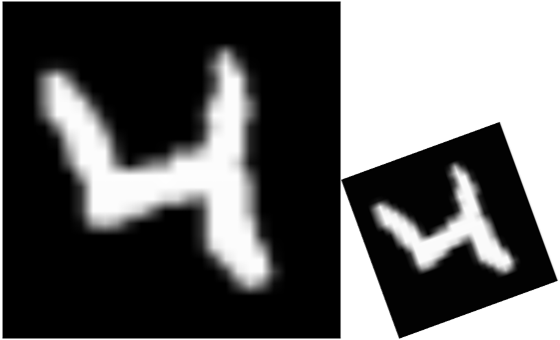

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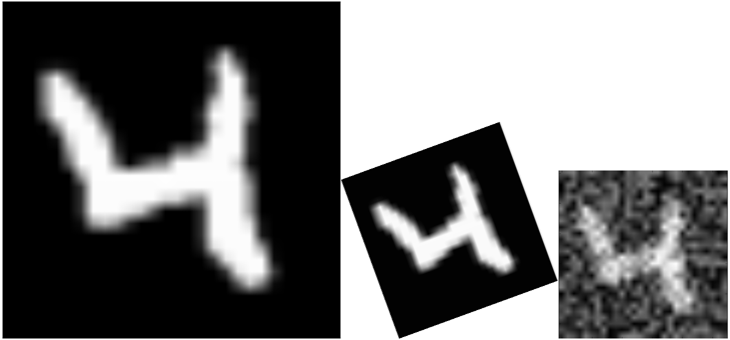

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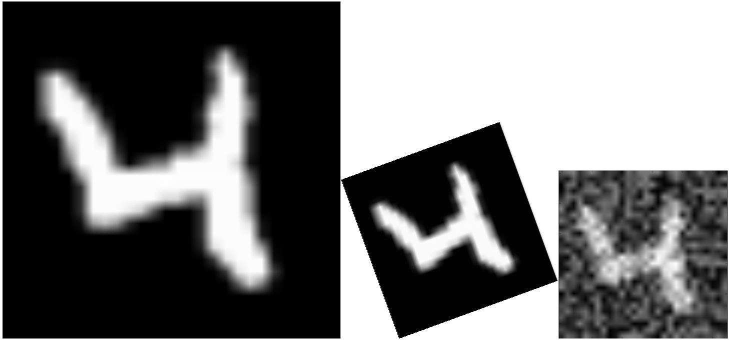

Sometimes also noise is added to the weights to favour ‘robust’ stationary points.

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Figure: "Extremely Deep" Network

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$$\Phi(x) = w_5\sigma(w_4\sigma(w_3\sigma(w_2\sigma(w_1 x + b_1) + b_2) + b_3) + b_4) + b_5$$

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🙁 bottom layers will learn *much* slower than top layers and not contribute to learning. Is depth a nuisance!?

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💡 Use activation function with 'large' gradient.

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ReLU

The Rectified Linear Unit is defined as

$$\mathsf{ReLU}(x) := \begin{cases} x & x > 0 \\ 0 & \text{else} \end{cases}$$

7 Convolutional Neural Networks


Is there a cat in this image?

Suppose we have a ‘cat-filter’ W

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C-S Inequality

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For any

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

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with equality if and only if X is parallel to a cat.

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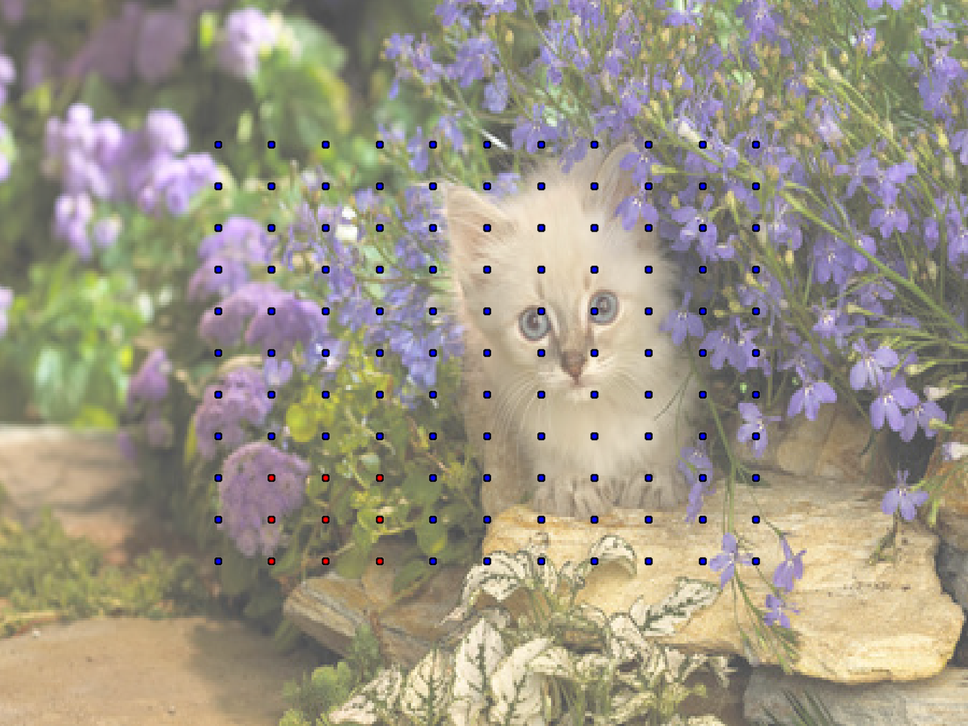
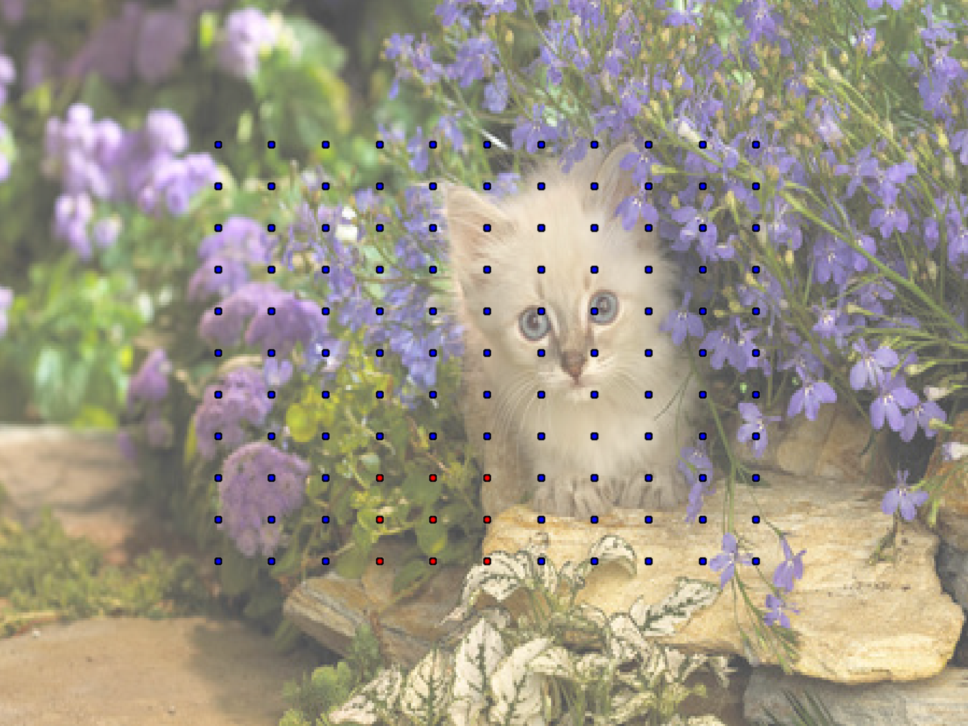
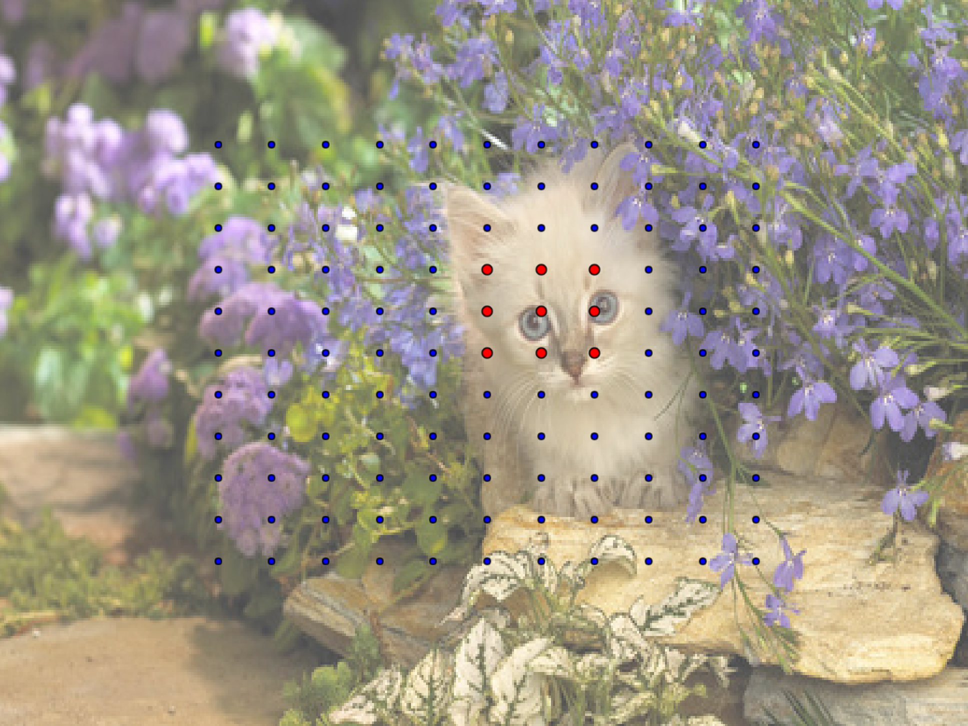
perform cat-test on all 3×3 image subpatches!







Convolution

Definition

Suppose that $X, Y \in \mathbb{R}^{n \times n}$. Then $Z = X * Y \in \mathbb{R}^{n \times n}$ is defined as

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 Efficient computation possible via FFT (or directly if X or Y are sparse)!

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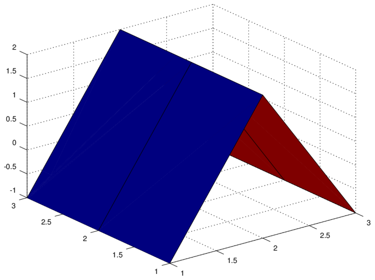

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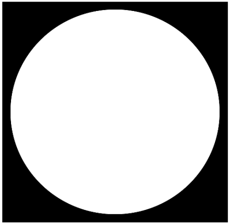

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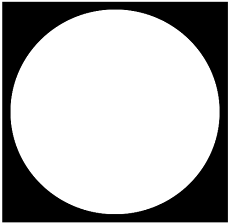

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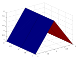

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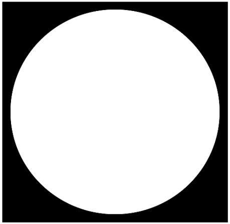
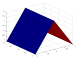
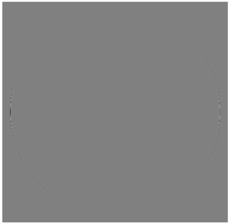


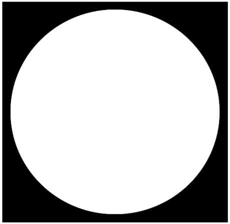
$*$



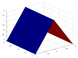
$=$



Example: Detecting Vertical Edges



*



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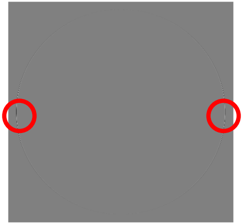

Introducing Convolutional Nodes

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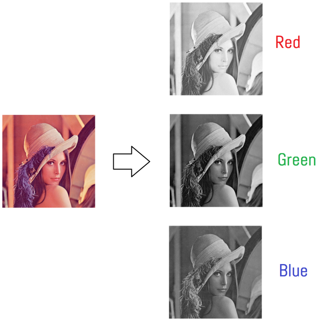

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- Given a filter $W \in \mathbb{R}^{F \times F \times S}$, where F is the *spatial extent* and a *bias* $b \in \mathbb{R}$, it computes a matrix

$$Z = W *_{12} X := \sum_{i=1}^{S} X[:,:,i] * W[:,:,i] + b.$$

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- A convolutional layer consists of K convolutional nodes $((W_i, b_i))_{i=1}^{K} \subset \mathbb{R}^{F \times F \times S} \times \mathbb{R}$ and produces as output a stack $Z \in \mathbb{R}^{n_1 \times n_2 \times K}$ via

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A convolutional layer can be written as a conventional neural network layer!

Activation Layers

The activation layer is defined in the same way as before, e.g., $Z \in \mathbb{R}^{n_1 \times n_2 \times K}$ is mapped to

$$A = \mathsf{ReLU}(Z)$$

where ReLU is applied component-wise.

Pooling Layers

- Reduce dimensionality after filtering.

Pooling Layers

Reduce dimensionality after filtering.

Definition

A *pooling operator* $\mathbf{R}$ acts layer-wise on a tensor $X \in \mathbb{R}^{n_1 \times n_2 \times S}$ to result in a tensor $\mathbf{R}(X) \in \mathbb{R}^{m_1 \times m_2 \times S}$, where $m_1 < n_1$ and $m_2 < n_2$.

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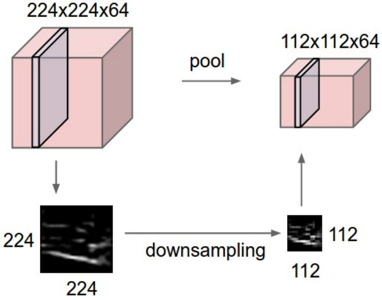

Downsampling

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💡 Reduce dimensionality after filtering.

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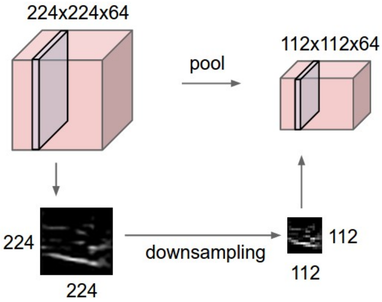

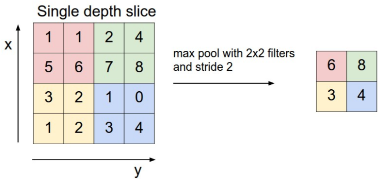

Max-pooling

Downsampling

Convolutional Neural Networks (CNNs)

Definition

A CNN with L layers consists of L iterative applications of a convolutional layer, followed by an activation layer, (possibly) followed by a pooling layer.

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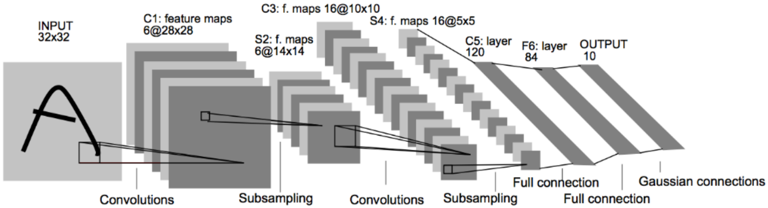

Figure: LeNet (1998, LeCun etal): the first successful CNN architecture, used for reading handwritten digits

Feature Extractor vs. Classifier












Feature Extractor vs. Classifier

Feature Extractor vs. Classifier

D. Trump

B. Sanders

A. Merkel

B. Johnson

```
x_image = tf.reshape(x, [-1, 28, 28, 1])
# First convolutional layer - maps one grayscale image to 32 feature maps.
W_conv1 = weight_variable([5, 5, 1, 32])
b_conv1 = bias_variable([32])
h_conv1 = tf.nn.relu(conv2d(x_image, W_conv1) + b_conv1)
# Pooling layer - downsamples by 2X.
h_pool1 = max_pool_2x2(h_conv1)
# Second convolutional layer -- maps 32 feature maps to 64.
W_conv2 = weight_variable([5, 5, 32, 64])
b_conv2 = bias_variable([64])
h_conv2 = tf.nn.relu(conv2d(h_pool1, W_conv2) + b_conv2)
# Second pooling layer.
h_pool2 = max_pool_2x2(h_conv2)
# Fully connected layer 1 -- after 2 round of downsampling, our 28x28 image
W_fc1 = weight_variable([7 * 7 * 64, 1024])
b_fc1 = bias_variable([1024])
h_pool2_flat = tf.reshape(h_pool2, [-1, 7*7*64])
h_fc1 = tf.nn.relu(tf.matmul(h_pool2_flat, W_fc1) + b_fc1)
# Map the 1024 features to 10 classes, one for each digit
W_fc2 = weight_variable([1024, 10])
b_fc2 = bias_variable([10])
y_conv = tf.matmul(h_fc1, W_fc2) + b_fc2
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Python Library *Tensor Flow*, developed by Google Brain, based on symbolic computational graphs `www.tensorflow.org`.

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b_co ... 32])
h_co ... 2d(x_image, W_conv1) + b_conv1)
# Po ... ples by 2X.
h_po ... conv1)
# Se ... yer -- maps 32 feature maps to 64.
W_co ... e([5, 5, 32, 64])
b_co ... 64])
h_co ... 2d(h_pool1,
# Se
h_po ... conv2)
# Fu ... -- after
W_fc1 ... 7 * 7 * 64, 1024])
b_fc1 ... 024])
h_po ... e(h_pool2, [-1, 7*7*64])
h_fc1 ... mul(h_pool2_flat, W_fc1) + b_fc1)
# Map ... o 10 classes, one for each digit
W_fc2 ... 1024, 10])
b_fc2 ... ])
y_con ... W_fc2) + b_fc2
```

```
step 15800, training accuracy 1
step 15900, training accuracy 1
step 16000, training accuracy 1
step 16100, training accuracy 1
step 16200, training accuracy 1
step 16300, training accuracy 1
step 16400, training accuracy 1
step 16500, training accuracy 1
step 16600, training accuracy 1
step 16700, training accuracy 1
step 16800, training accuracy 1
step 16900, training accuracy 1
step 17000, training accuracy 1
step 17100, training accuracy 1
step 17200, training accuracy 1
step 17300, training accuracy 1
step 17400, training accuracy 1
step 17500, training accuracy 1
step 17600, training accuracy 1
step 17700, training accuracy 1
step 17800, training accuracy 1
step 17900, training accuracy 1
step 18000, training accuracy 1
step 18100, training accuracy 1
step 18200, training accuracy 1
step 18300, training accuracy 1
step 18400, training accuracy 1
step 18500, training accuracy 1
step 18600, training accuracy 1
step 18700, training accuracy 1
step 18800, training accuracy 1
step 18900, training accuracy 1
step 19000, training accuracy 1
step 19100, training accuracy 1
step 19200, training accuracy 1
step 19300, training accuracy 1
step 19400, training accuracy 1
step 19500, training accuracy 1
step 19600, training accuracy 1
step 19700, training accuracy 1
step 19800, training accuracy 1
step 19900, training accuracy 1
test accuracy 0.9934
```

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8 What I didn't tell you

1. Data structures & algorithms for efficient deep learning (computational graphs, automatic differentiation, adaptive learning rate, hardware, ...)

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5. Recurrent Neural Networks

Questions?